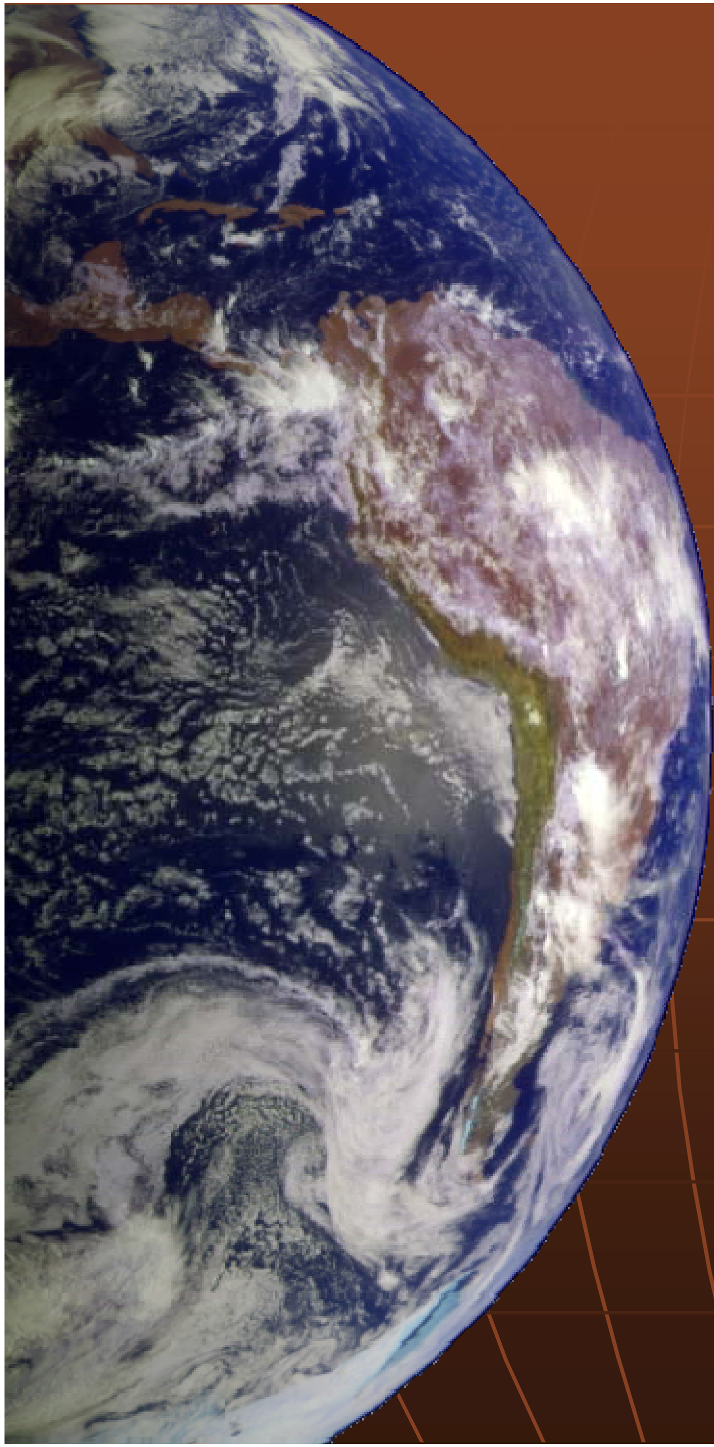


UT-AUSTIN EDGER FORUM

2D Seismic modeling based on SEM in Triangular mesh (TSEM)

Tao Liu (Peking University)
Mrinal Sen (UT at Austin)

THE UNIVERSITY OF TEXAS AT AUSTIN
JACKSON
SCHOOL OF GEOSCIENCES

Outline

- Motivation
- Theory
- Numerical examples
- -- Regular boundary
- -- Topography problem
- Discussion

UT-AUSTIN EDGER FORUM

THE UNIVERSITY OF TEXAS AT AUSTIN JACKSON SCHOOL OF GEOSCIENCES

Motivation

- Finite element method has the advantage of dealing with irregular boundary condition for seismic modeling.
- It demands much more computational resources since it requires solving a matrix equation (mass matrix).
- Spectral element is supposed to be able to solve the wave propagation efficiently with a diagonal mass matrix and promising accuracy.

Motivation

- However most of the spectral element methods are implemented with quadrilateral or hexahedral meshes, due to the high accuracy quadrature rule and convenience of obtaining the interpolation polynomials.

- In contrast, if we want to simulate the wave propagation with irregular boundary, triangular mesh is preferable – this introduces additional difficulties.

Theory for 2D problems

UT-AUSTIN EDGER FORUM

- For 2D acoustic wave propagation, the wave equation has the following form

$$\mathbf{M}\frac{\mathrm{d}^2\mathbf{U}(t)}{\mathrm{d}t^2} - \mathbf{KU}(t) = 0$$

- where the mass and stiffness matrices are given by

$$\mathbf{M}_{p,q} = \iint_R \lambda_{l,m}(x,y)\lambda_{l_0,m_0}(x,y)dxdy \qquad p:->(l,m)$$

$$\mathbf{K}_{p,q} = \iint_R c^2\left(\frac{\partial\lambda_{l,m}}{\partial x}\frac{\partial\lambda_{l_0,m_0}}{\partial x} + \frac{\partial\lambda_{l,m}}{\partial y}\frac{\partial\lambda_{l_0,m_0}}{\partial y}\right)dxdy \quad q:->(l_0,m_0)$$

THE UNIVERSITY OF TEXAS AT AUSTIN JACKSON SCHOOL OF GEOSCIENCES

Difficulties in triangle

- In SEM, to get a diagonal mass matrix, we have to employ the Lagrange polynomial and quadrature rule.
- Unfortunately, in a triangular mesh, it is very hard for us to find out an explicit expression for Lagrange polynomial, and there are no GLL points which have been popularly employed in quadrilateral mesh.
- Some optimal points are obtained to replace the GLL points, and the best chosen basis function is introduced, to construct a Vandermonde system to get the Lagrange polynomial.

Optimal nodal position in a triangle

- **Electro static points (Hesthaven,1998)**

  Employing minimization of an electrostatic potential to determine the nodal set of points, and the nodes along the edge are the same as GLL points in a quadrilateral element.

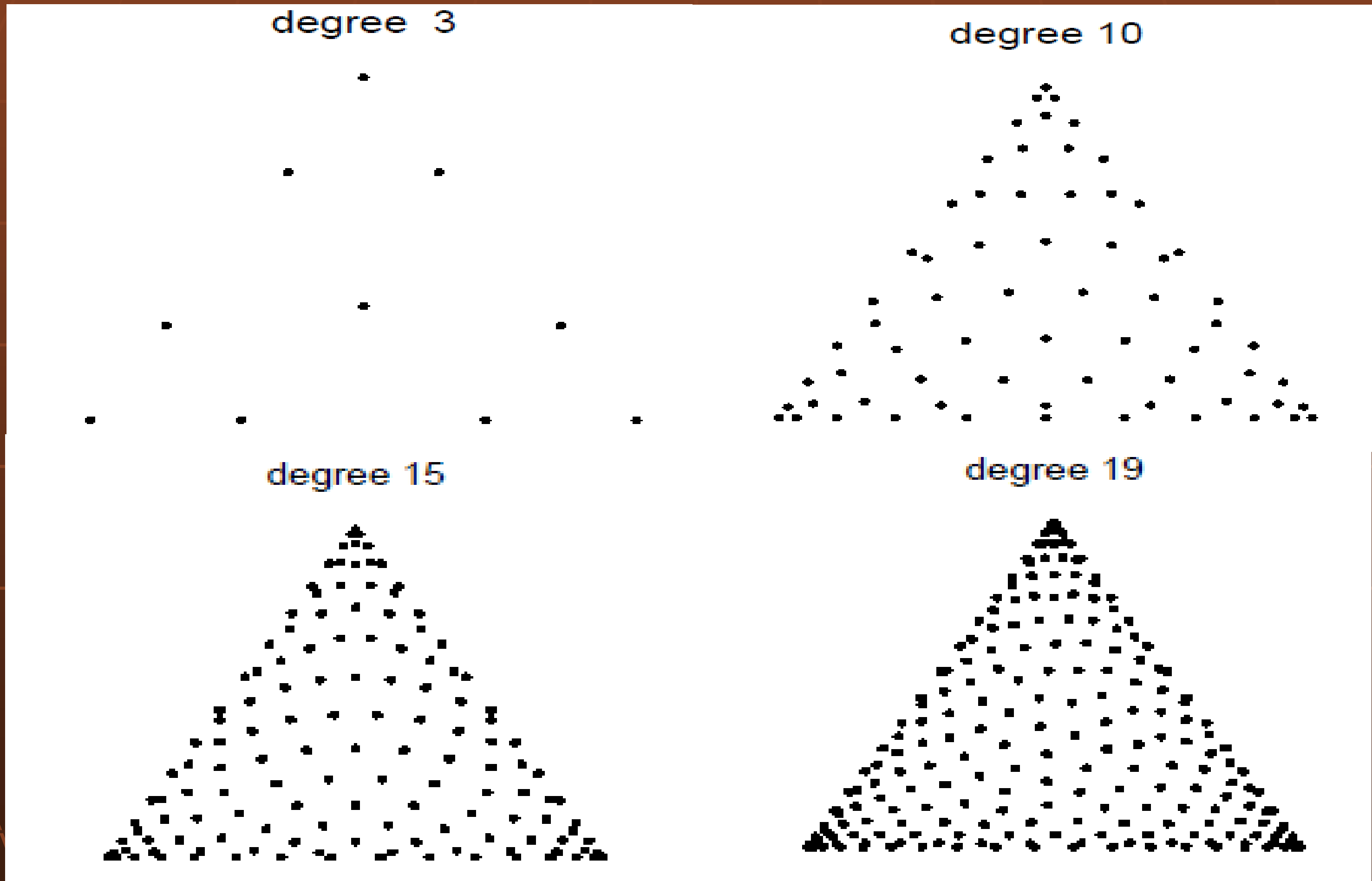
- **Fekete Points (Bos, 1983; Taylor et al.,2000)**

  points are obtained by minimizing the Legesgue constant, which reveal how close the polynomial approaches the function. The nodes along the edge are the identical to the GLL points as well.

- **L2 Points (Chen and Babuska,1995)**

  points are obtained with an L2 norm optimal Legesgue constant, but the nodes along the edge are not identified with the GLL points.

Fekete points



(Taylor et al, 2000)

Lagrange polynomial

- **The Lagrange polynomial has the expansion of**

$$\lambda_i(\xi,\eta)=\sum_{j=1}^{N}c_j^i\cdot\phi_j(\xi,\eta) \qquad \lambda_i(\xi_j,\eta_j)=\delta_{ij}$$

- **Which can be transformed to the form of**

$$\mathbf{V}\cdot\mathbf{c}_i=\mathbf{e}_i$$

where **V** is the $N\times N$ **generalized Vandermonde** matrix

$$V_{kj}=\phi_j(\xi_k,\eta_k) \quad \mathbf{c}_i=(c_1^i,\ldots,c_j^i,\ldots c_N^i)^{\mathrm{T}} \quad \mathbf{e}_i=(0,\ldots,0,1,0,\ldots 0)^{\mathrm{T}}$$

Lagrange polynomial

UT-AUSTIN EDGER FORUM

- More general situation

$$\begin{bmatrix} \phi_1(\xi_1,\eta_1) & \cdots & \phi_i(\xi_1,\eta_1) & \cdots & \phi_N(\xi_1,\eta_1) \\ \vdots & & \vdots & & \vdots \\ \phi_1(\xi_j,\eta_j) & \cdots & \phi_i(\xi_j,\eta_j) & \cdots & \phi_N(\xi_j,\eta_j) \\ \vdots & & \vdots & & \vdots \\ \phi_1(\xi_N,\eta_N) & \cdots & \phi_i(\xi_N,\eta_N) & \cdots & \phi_N(\xi_N,\eta_N) \end{bmatrix} \begin{bmatrix} c_1^1 & \dots & c_1^i & \dots & c_1^N \\ \vdots & & \vdots & & \vdots \\ c_j^1 & \dots & c_j^i & \dots & c_j^N \\ \vdots & & \vdots & & \vdots \\ c_N^1 & \dots & c_N^i & \dots & c_N^N \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

- Given the basis function and after obtaining the grid positions, we can compute out the coefficients c^i and then get the Lagrange polynomial.

THE UNIVERSITY OF TEXAS AT AUSTIN
JACKSON
SCHOOL OF GEOSCIENCES

The basis function

- The most popular basis function for triangle is **Dubiner polynomial (Dubiner, 1991)**.
- Dubiner polynomial is obtained from wrapped tensor product in the collapsed coordinate, which is an orthogonal one.

$$\phi_{pq}(r,s) = \psi_p^a(\eta_1)\psi_{pq}^b(\eta_2) \qquad \eta_1 = \frac{2(1+r)}{1-s} - 1, \quad \eta_2 = s$$

where $\begin{cases} \psi_p^a(z) = P_p^{0,0}(z) \\ \psi_{pq}^b(z) = (\frac{1-z}{2})^p P_p^{2p+1,0}(z) \end{cases}$

Coordinate transform

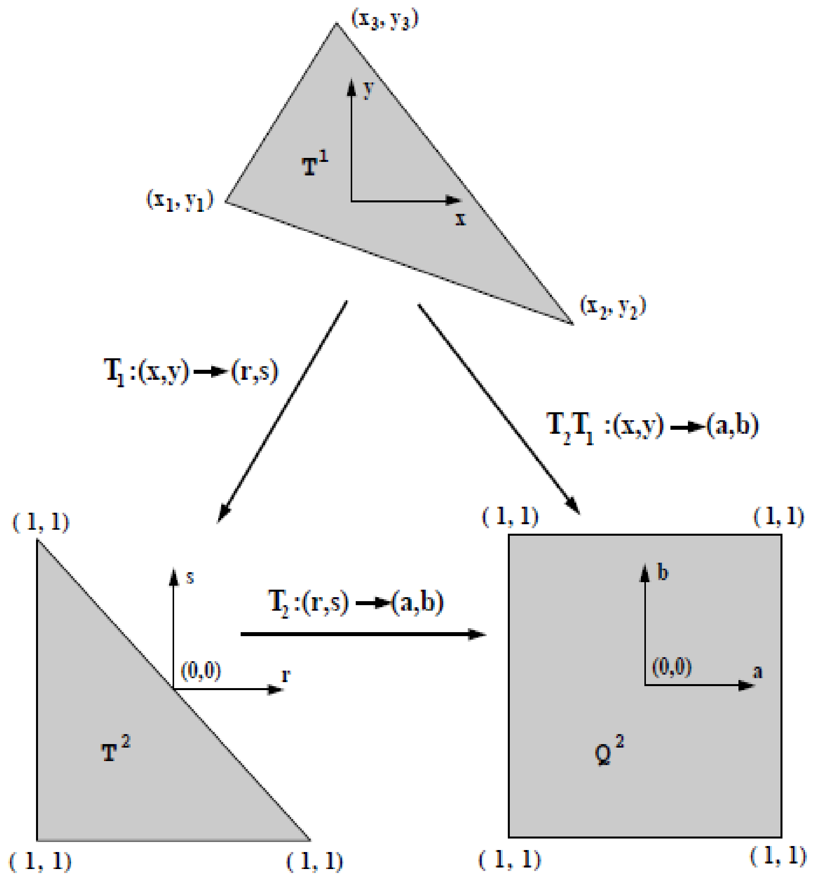


FIG. 2. *Transforms between the arbitrary triangle T^1, standard triangle T^2, and standard quadrilateral Q^2.*

Arbitrary oriented coordinate (x,y)

$$T^2 = \{(r,s)| -1 \le r,s; r+s \le 0\}$$

(Standard element)

$$Q^2 = \{(a,b)| -1 \le a,b \le 1\}$$

(Standard quadrilateral)

Collapsed coordinate system

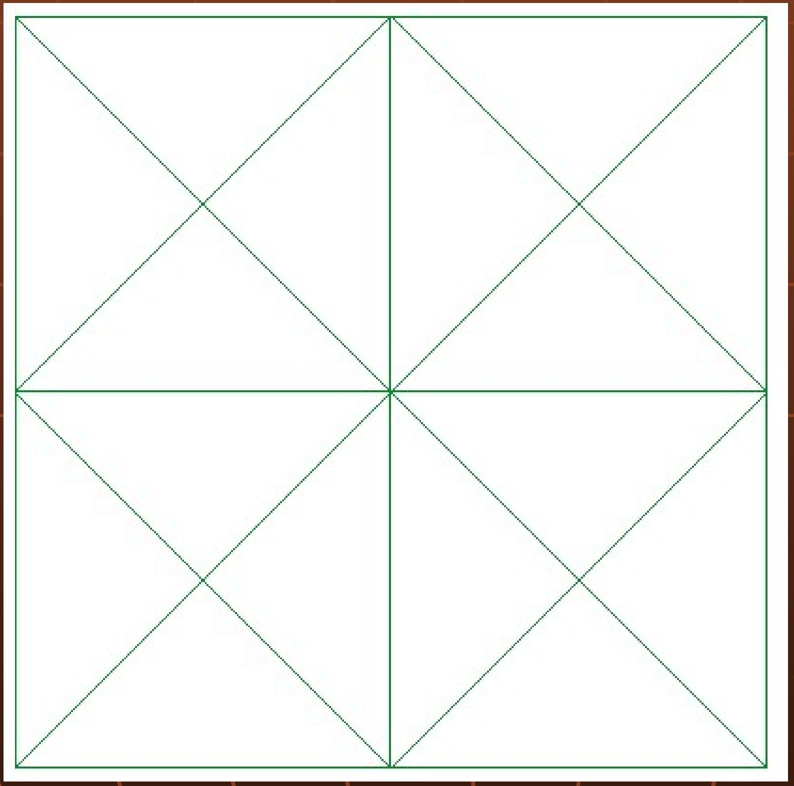
(Warburton et al, 2006)

UT-AUSTIN EDGER FORUM

THE UNIVERSITY OF TEXAS AT AUSTIN JACKSON SCHOOL OF GEOSCIENCES

Numerical experiments

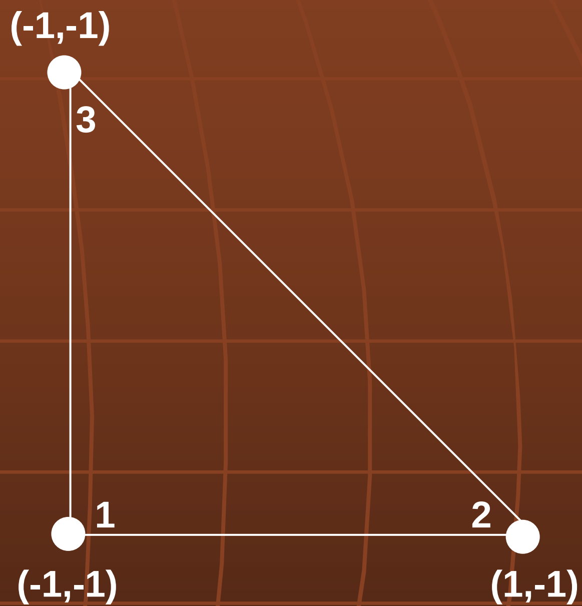
We consider the homogeneous medium with regular boundary



Triangular meshes generated by GID

1st order TSEM

$$V^{-1} = \begin{bmatrix} 0.333 & 0.333 & 0.333 \\ -0.1667 & -0.1667 & 0.333 \\ -0.50 & 0.50 & 0 \end{bmatrix}$$



$$\omega_i = 2V_{1i}^{-1} \Rightarrow \omega_1 = \omega_2 = \omega_3 = 0.6667$$

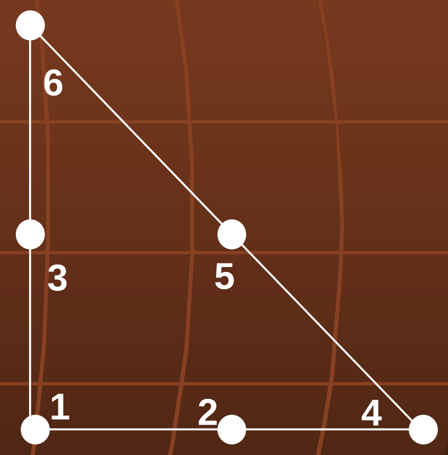
$$\begin{cases} \psi_1(r,s) = 0.333 - 0.0833 \cdot (1+3s) - 0.25 \cdot (1+2r+s) \\ \psi_2(r,s) = 0.333 - 0.0833 \cdot (1+3s) + 0.25 \cdot (1+2r+s) \\ \psi_3(r,s) = 0.333 + 0.1667 \cdot (1+3s) \end{cases}$$

2nd order TSEM

The inversion of Vandermonde matrix

$$V^{-1} =$$

$$\begin{bmatrix} 0 & 0.3333 & 0.3333 & 0 & 0.3333 & 0 \\ -0.1000 & -0.2667 & 0.1333 & -0.1000 & 0.1333 & 0.2000 \\ 0.0667 & 0.0667 & -0.2000 & 0.0667 & -0.2000 & 0.2000 \\ -0.3000 & 0 & -0.4000 & 0.3000 & 0.4000 & 0 \\ 0.2000 & 0 & -0.4000 & -0.2000 & 0.4000 & 0 \\ 0.3333 & -0.6667 & 0 & 0.3333 & 0 & 0 \end{bmatrix}$$



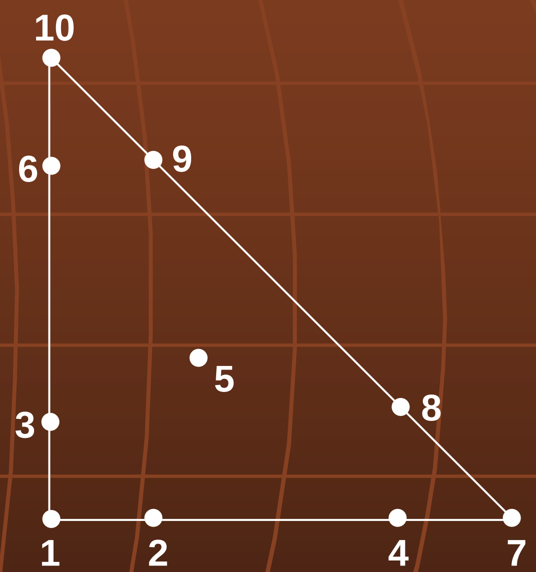
$$\omega_i = 2V_{1i}^{-1} \Rightarrow \omega_1 = \omega_4 = \omega_6 = 0$$

This configuration is not eligible for quadrature rule.

3rd order TSEM

Weight factors are positive

$$\begin{cases} \omega_1 = \omega_7 = \omega_{10} = 0.0333 \\ \omega_2 = \omega_3 = \omega_4 = \omega_6 = \omega_8 = \omega_9 = 0.1667 \\ \omega_5 = 0.9 \end{cases}$$



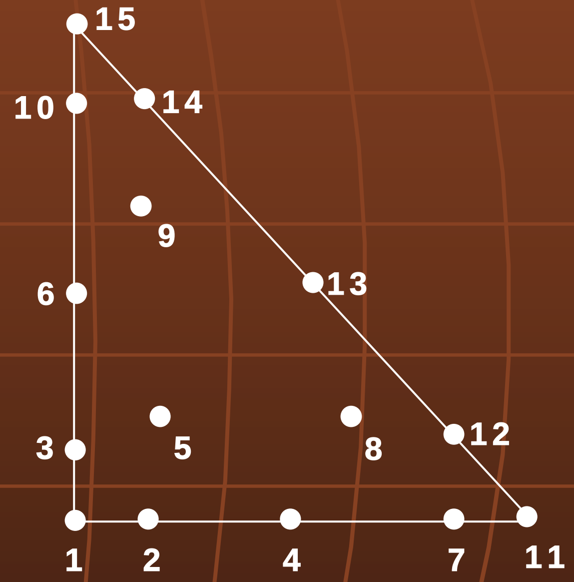
That's eligible for simulating the wave propagation

THE UNIVERSITY OF TEXAS AT AUSTIN
JACKSON
SCHOOL OF GEOSCIENCES

4th order TSEM

- Some of the weight factors are negative

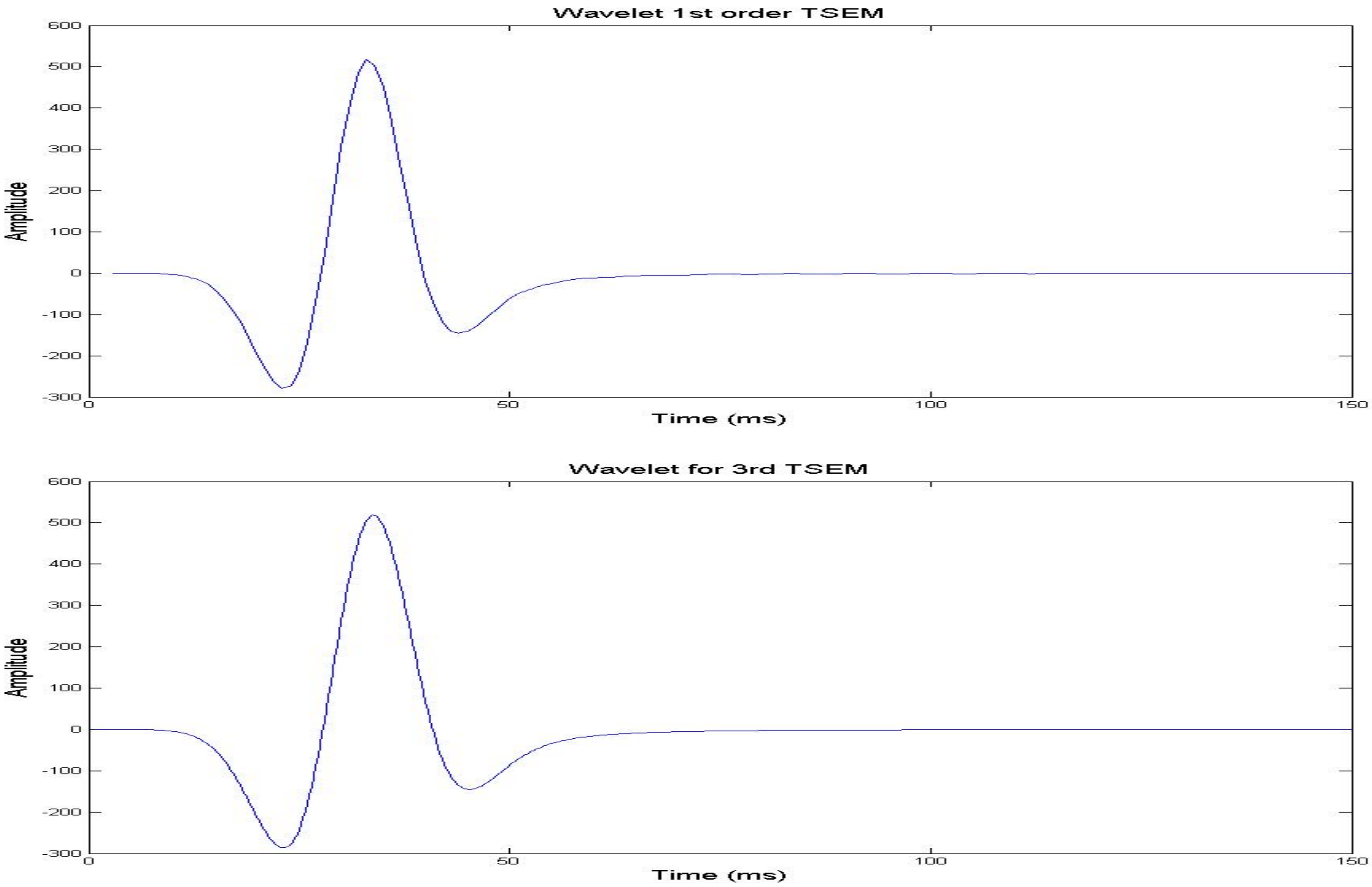
$$\begin{cases} \omega_1 = -0.0174 \\ \omega_{11} = -0.0174 \\ \omega_{15} = -0.0174 \end{cases}$$



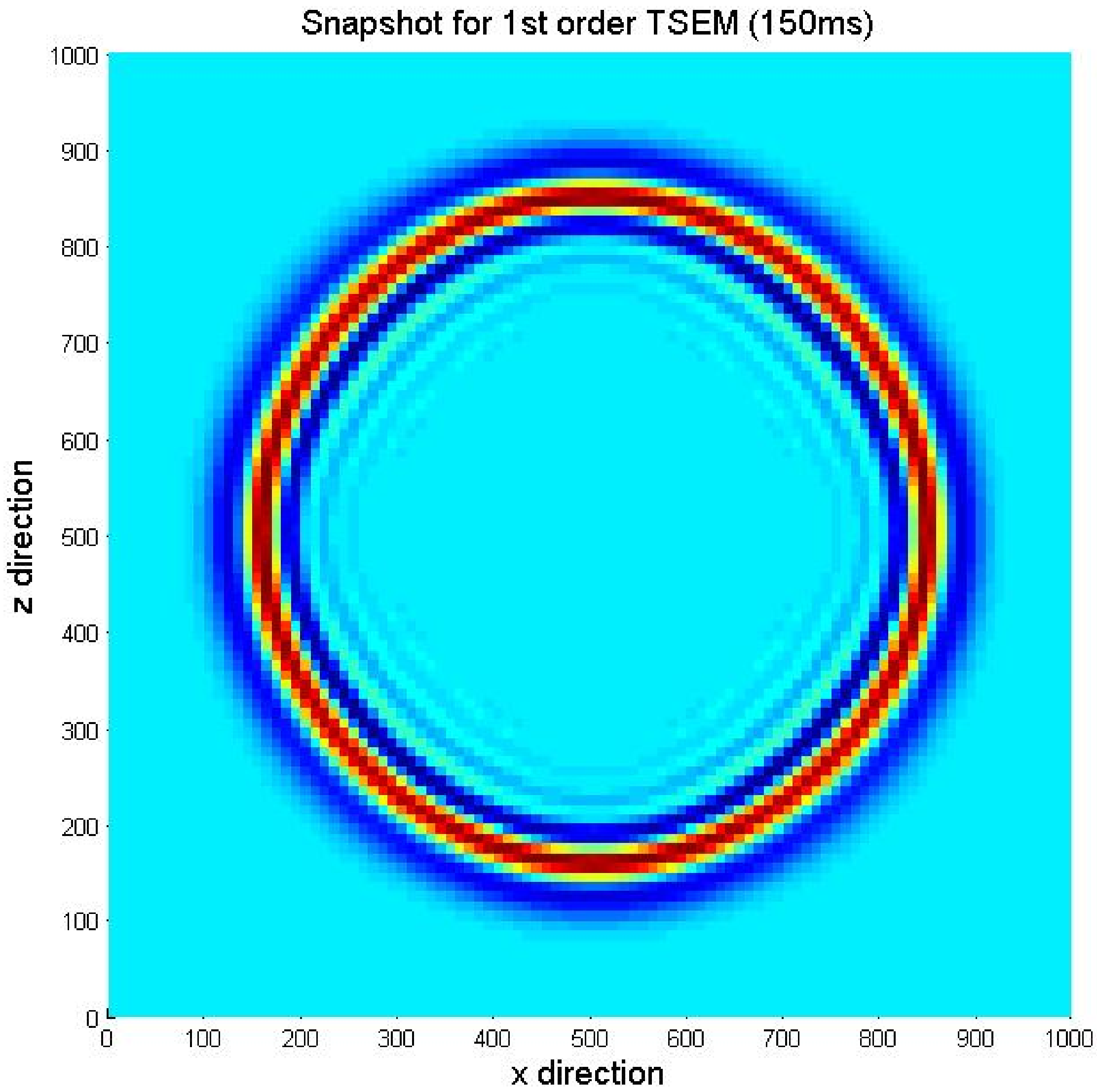
- As a result, the fourth order Fekete points are not eligible for seismic simulation.

THE UNIVERSITY OF TEXAS AT AUSTIN
JACKSON
SCHOOL OF GEOSCIENCES

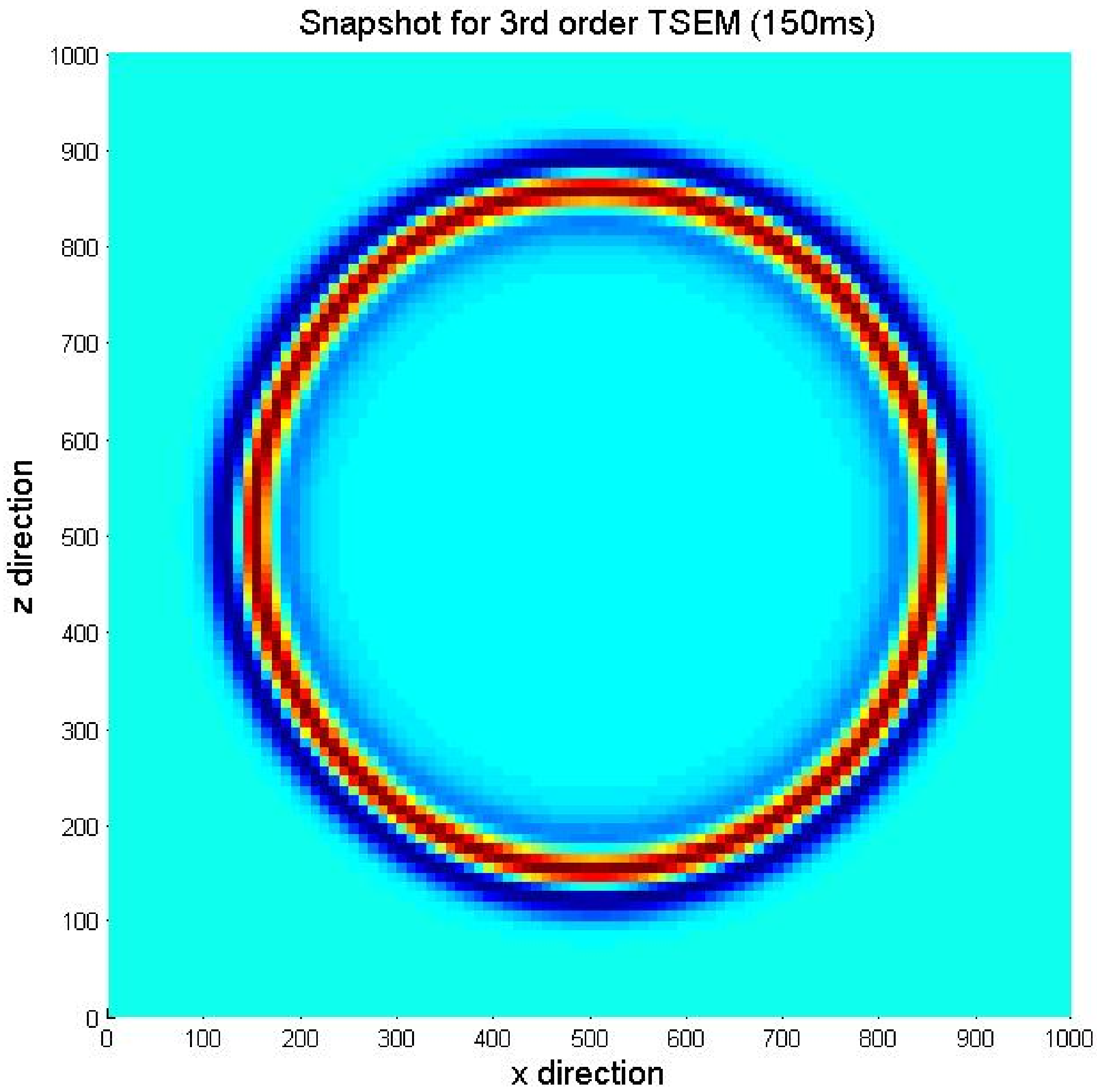
Dispersion behavior (40Hz)



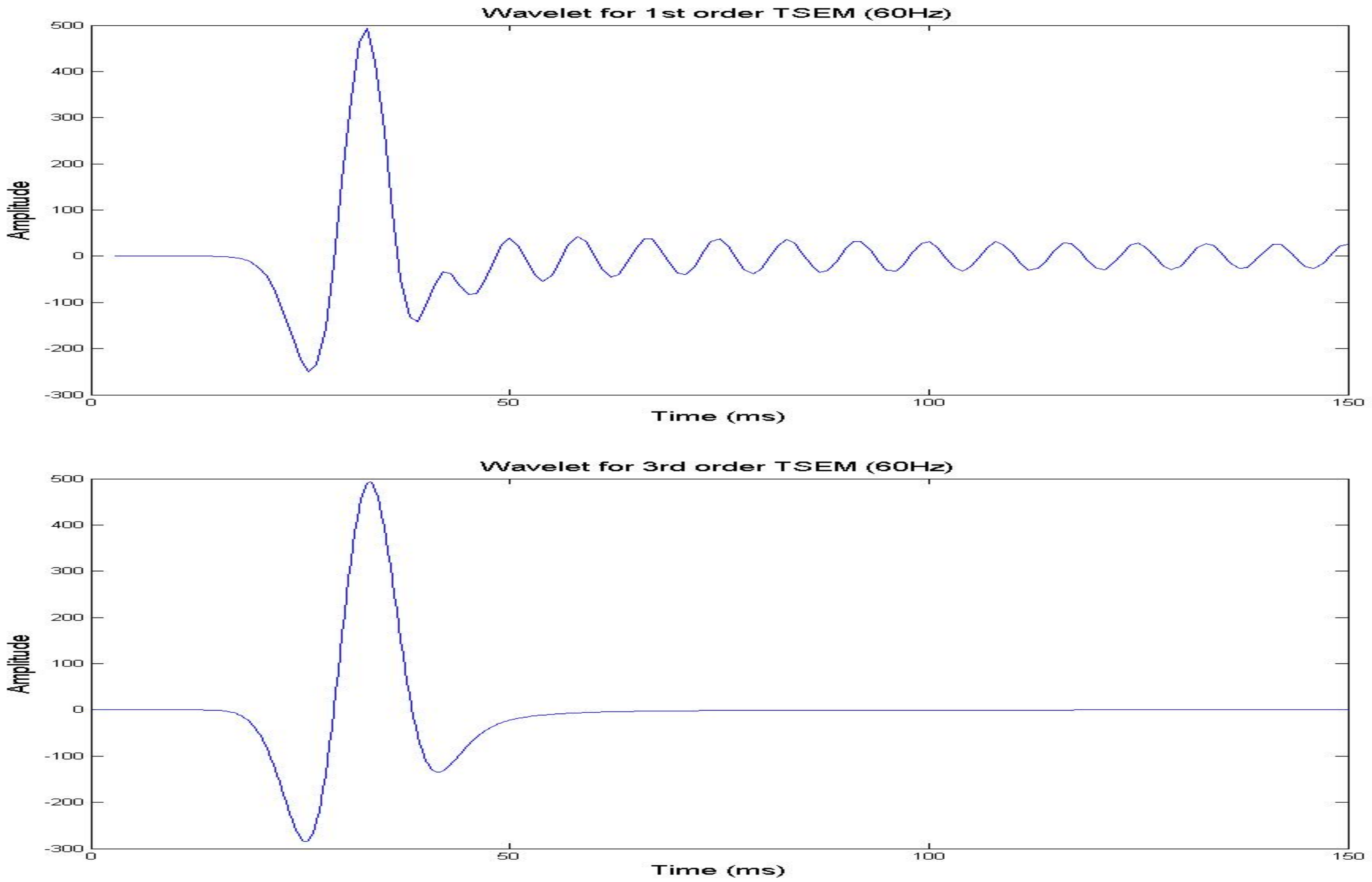
Snapshot for 1st order (40Hz)



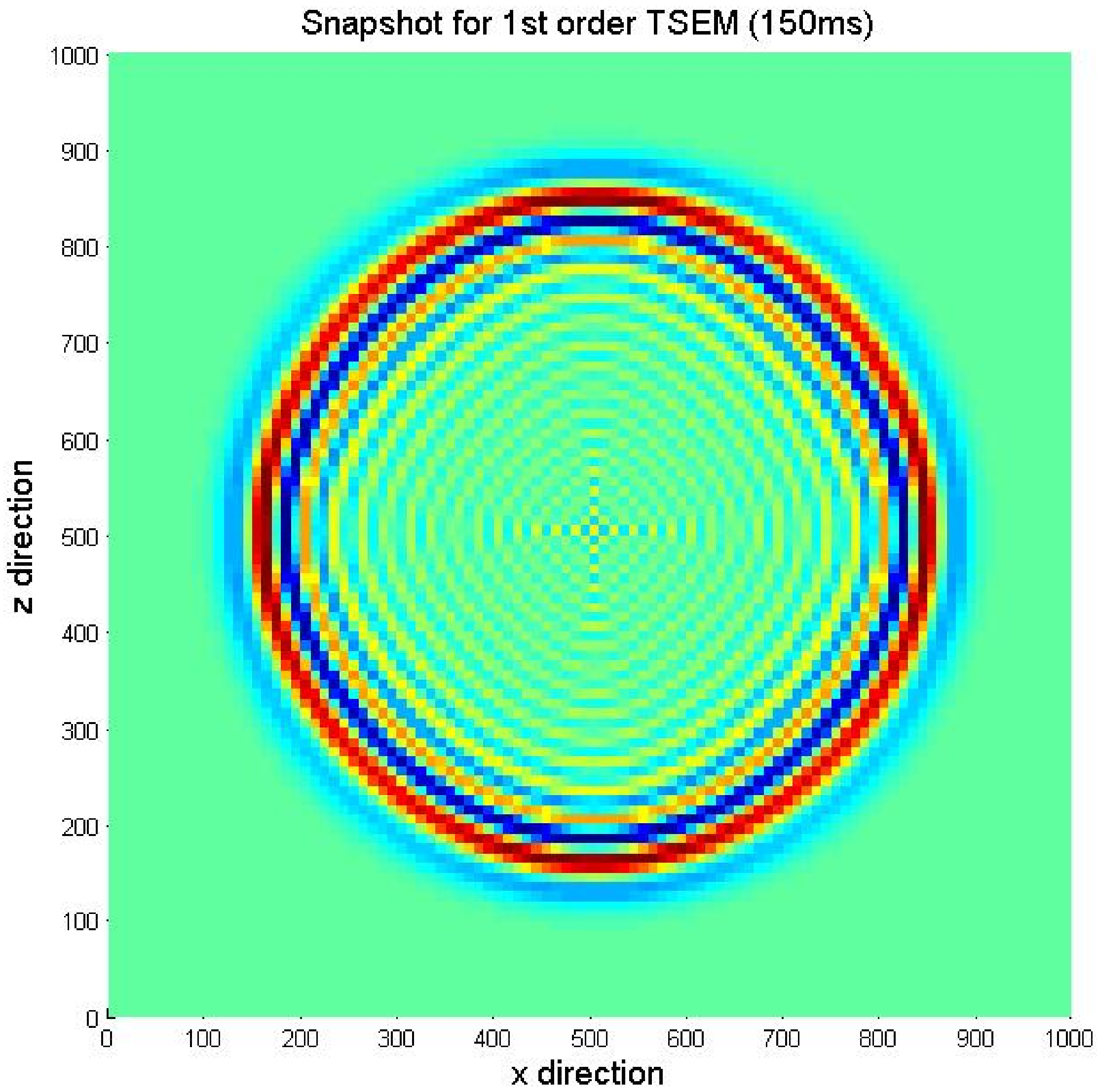
Snapshot for 3rd order (40Hz)



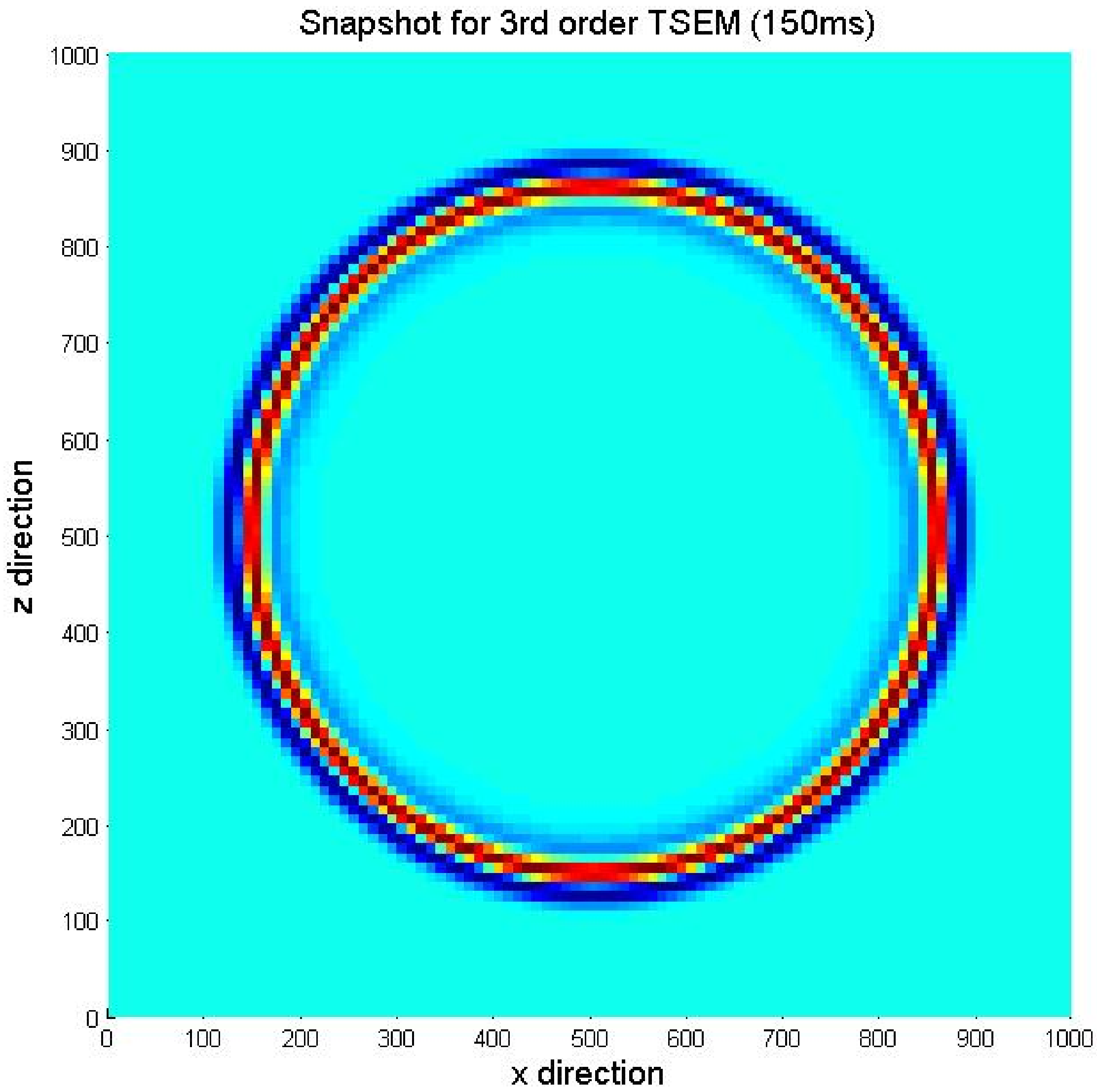
Dispersion Behavior (60Hz)



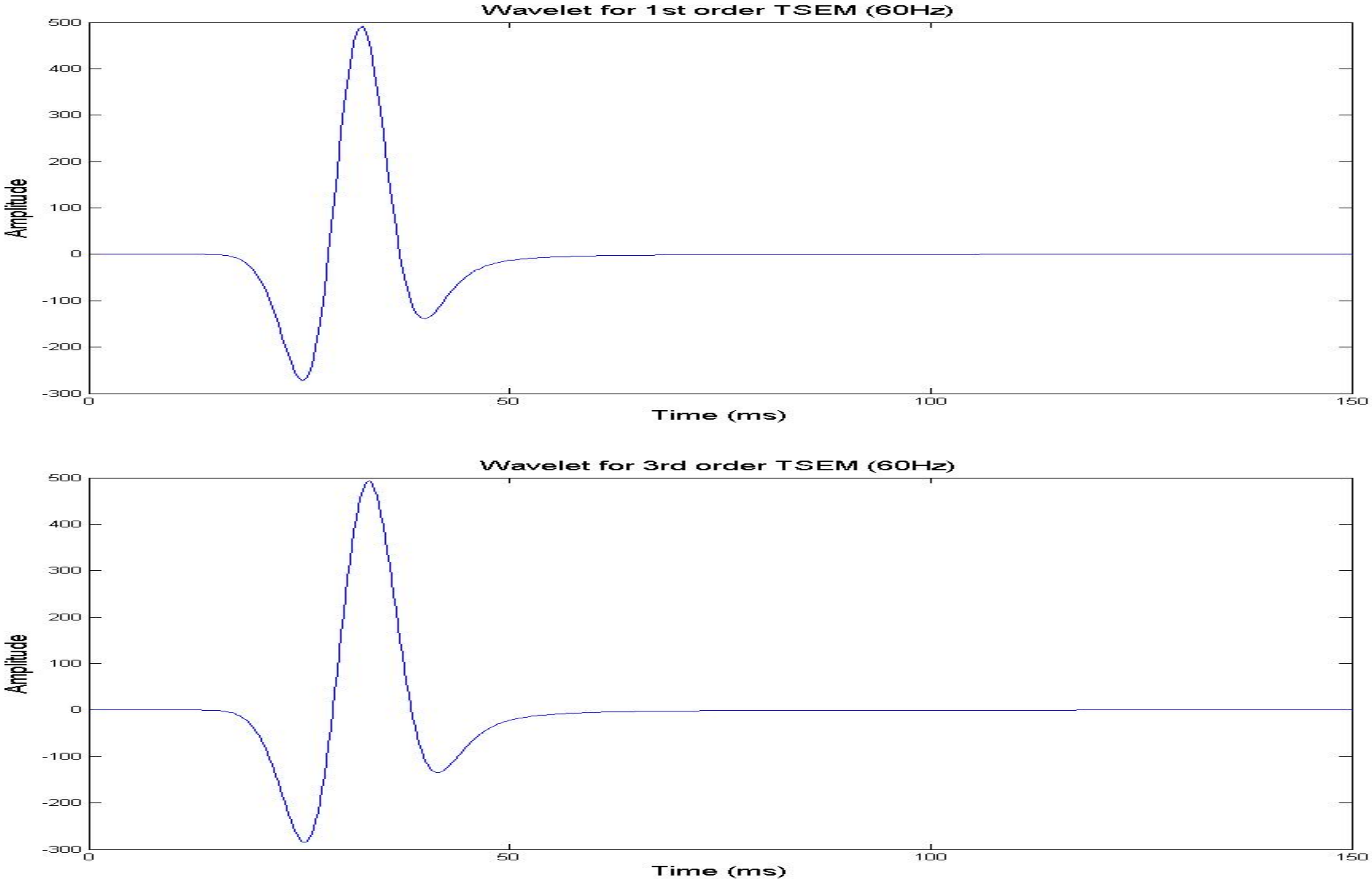
Snapshot for 1st order (60Hz)



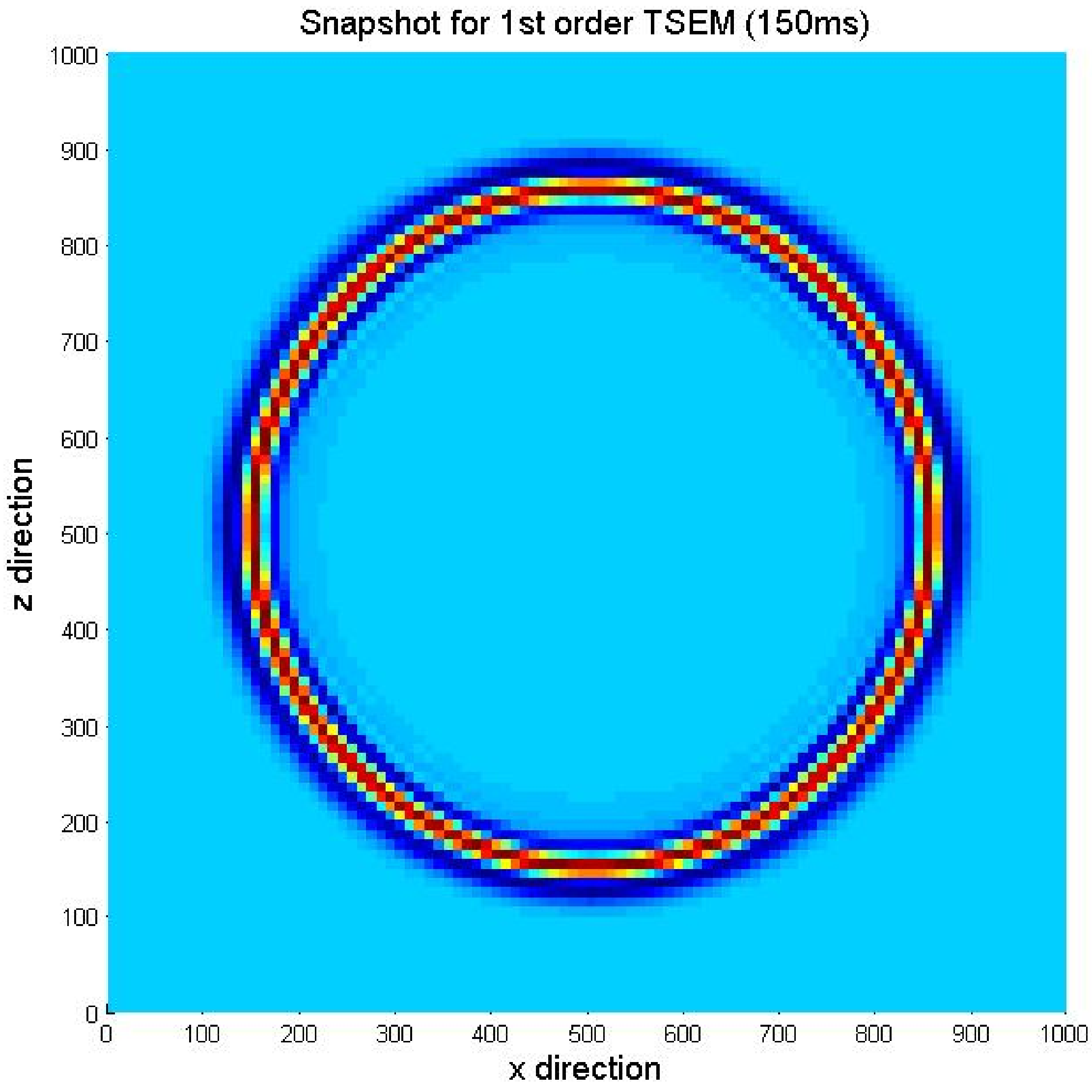
Snapshot for 3rd order (60Hz)



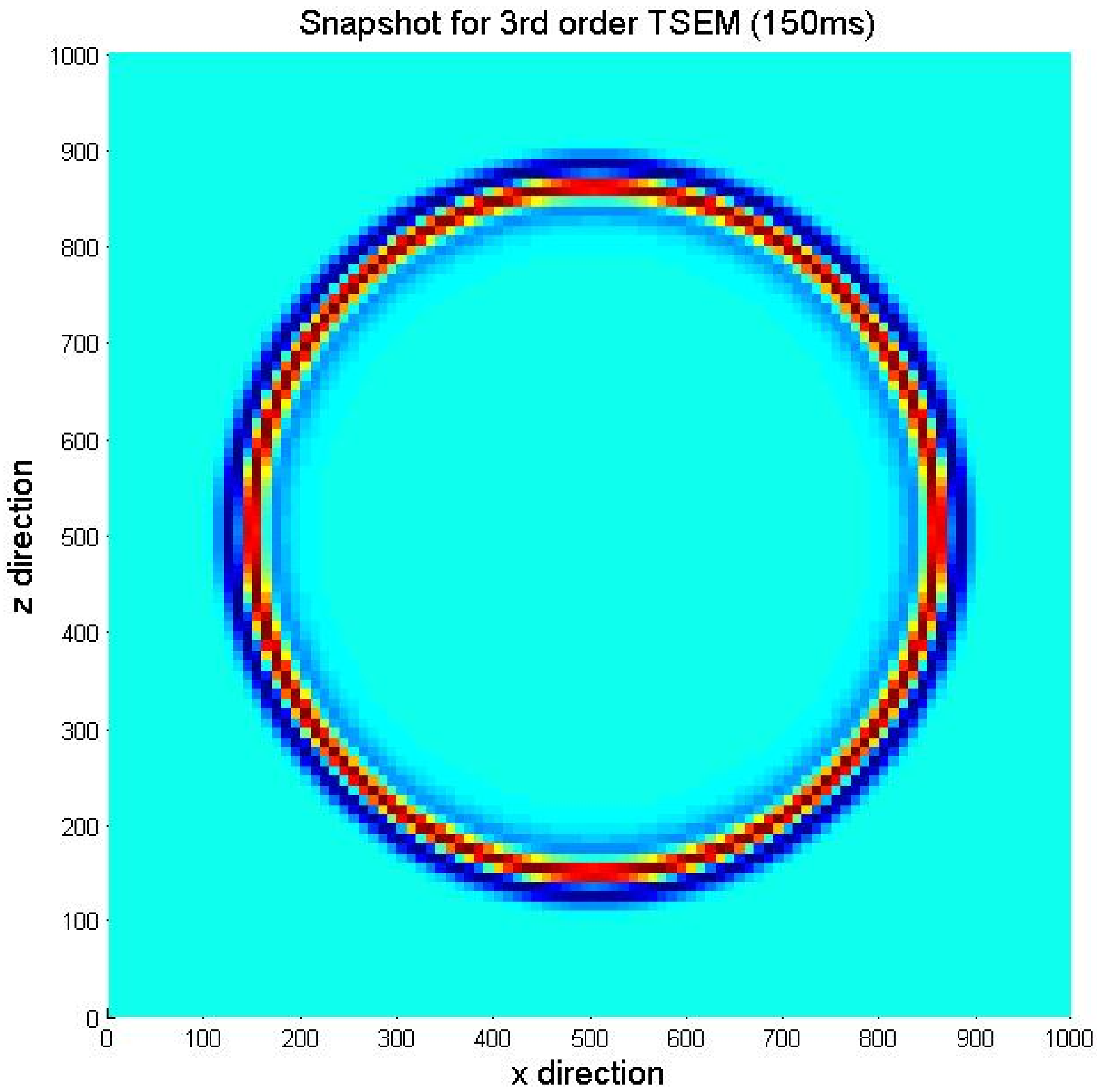
Employ smaller (half) size (60Hz)



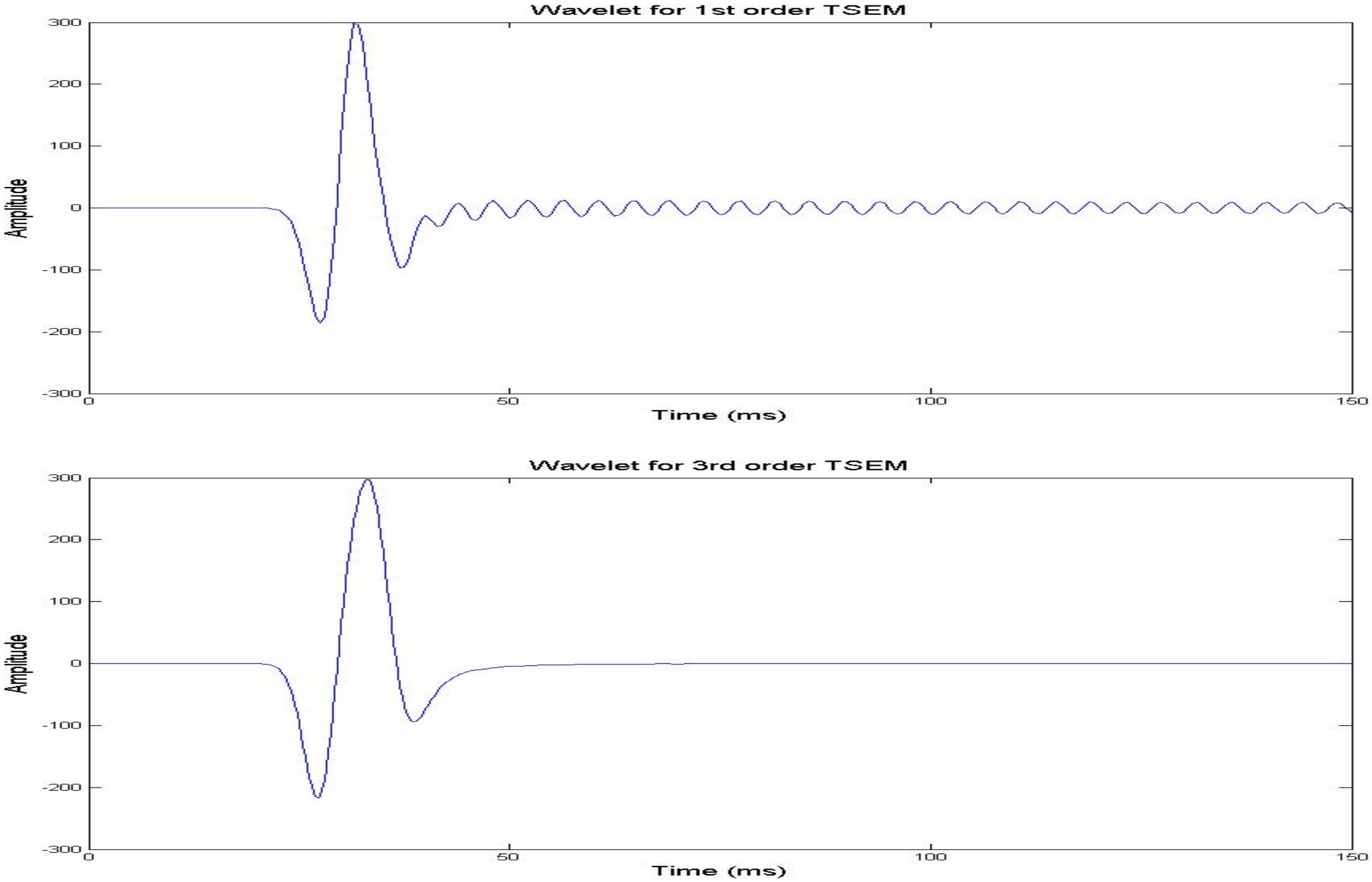
Snapshot for 1st order (60Hz)



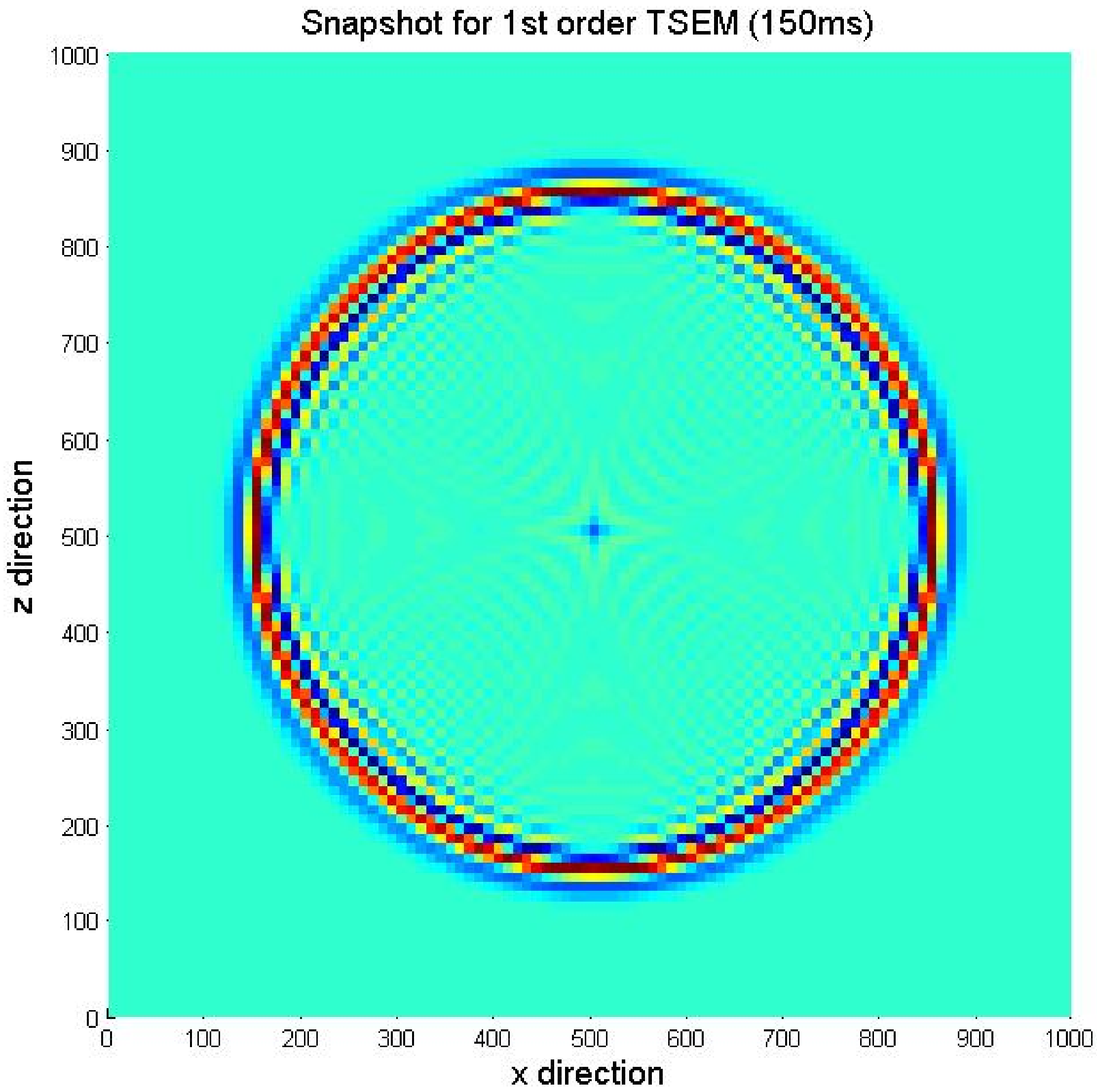
Snapshot for 3rd order (60Hz)



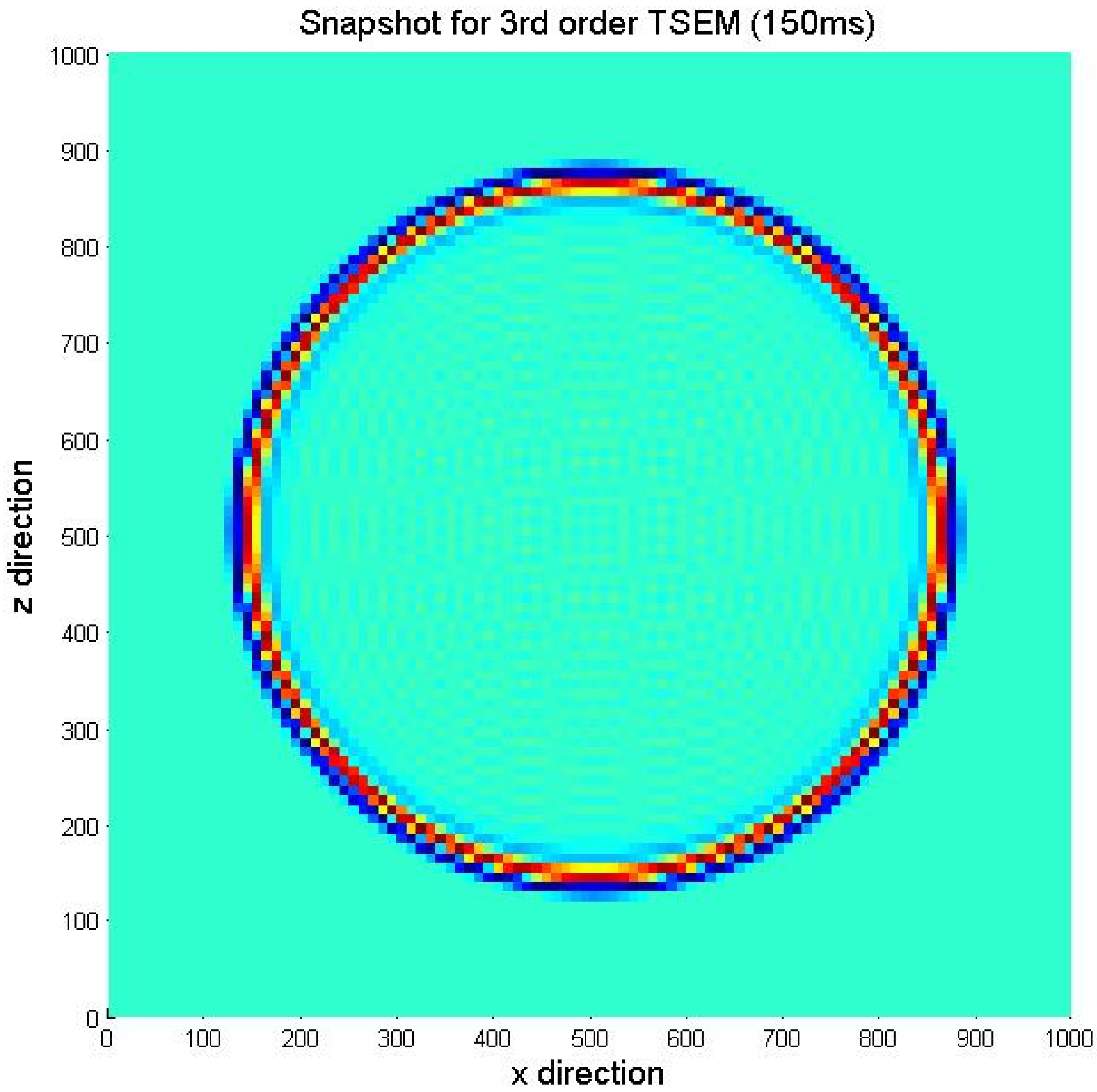
Employ smaller (half) size (100Hz)



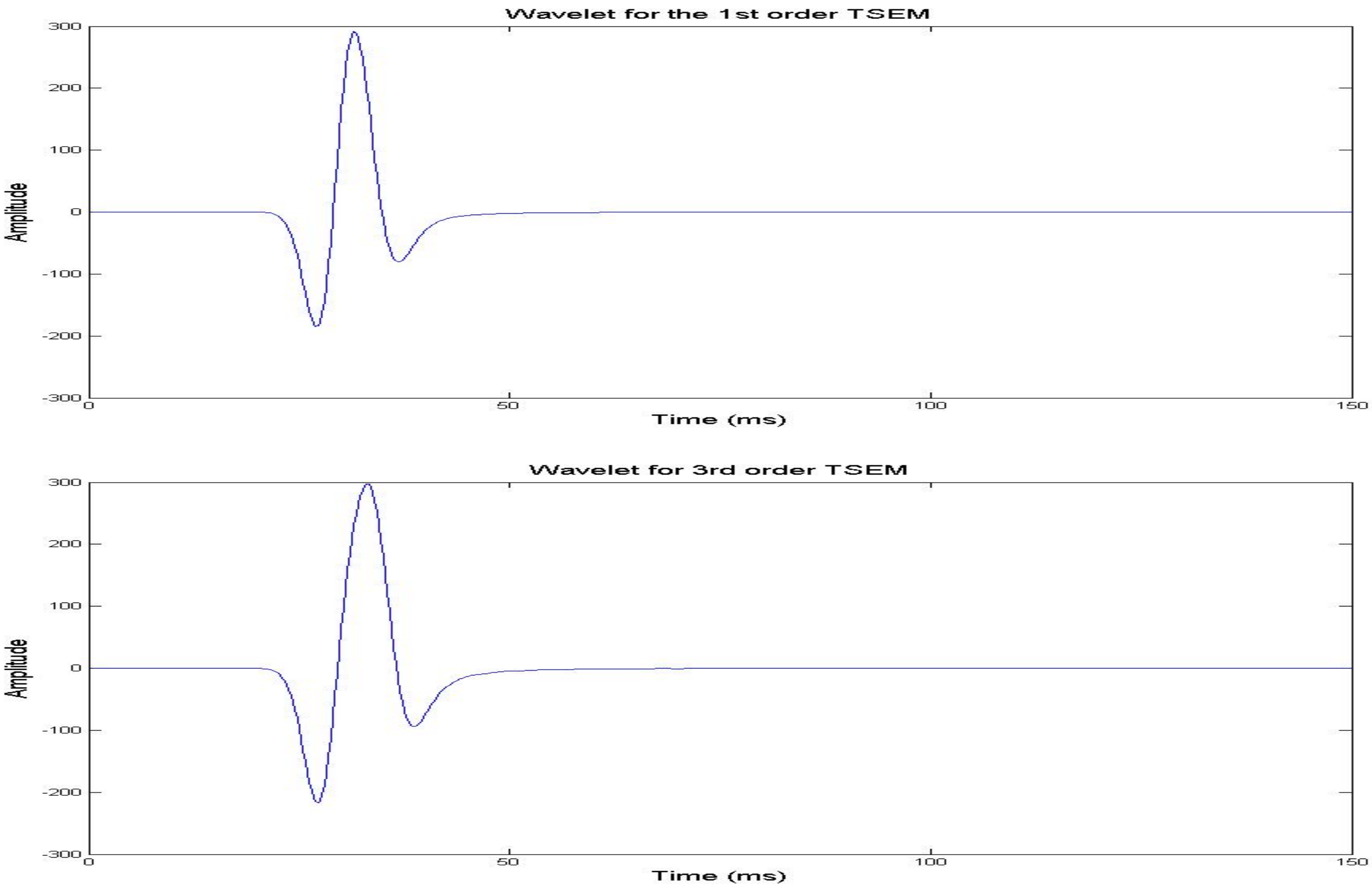
Snapshot for 1st order (100Hz)



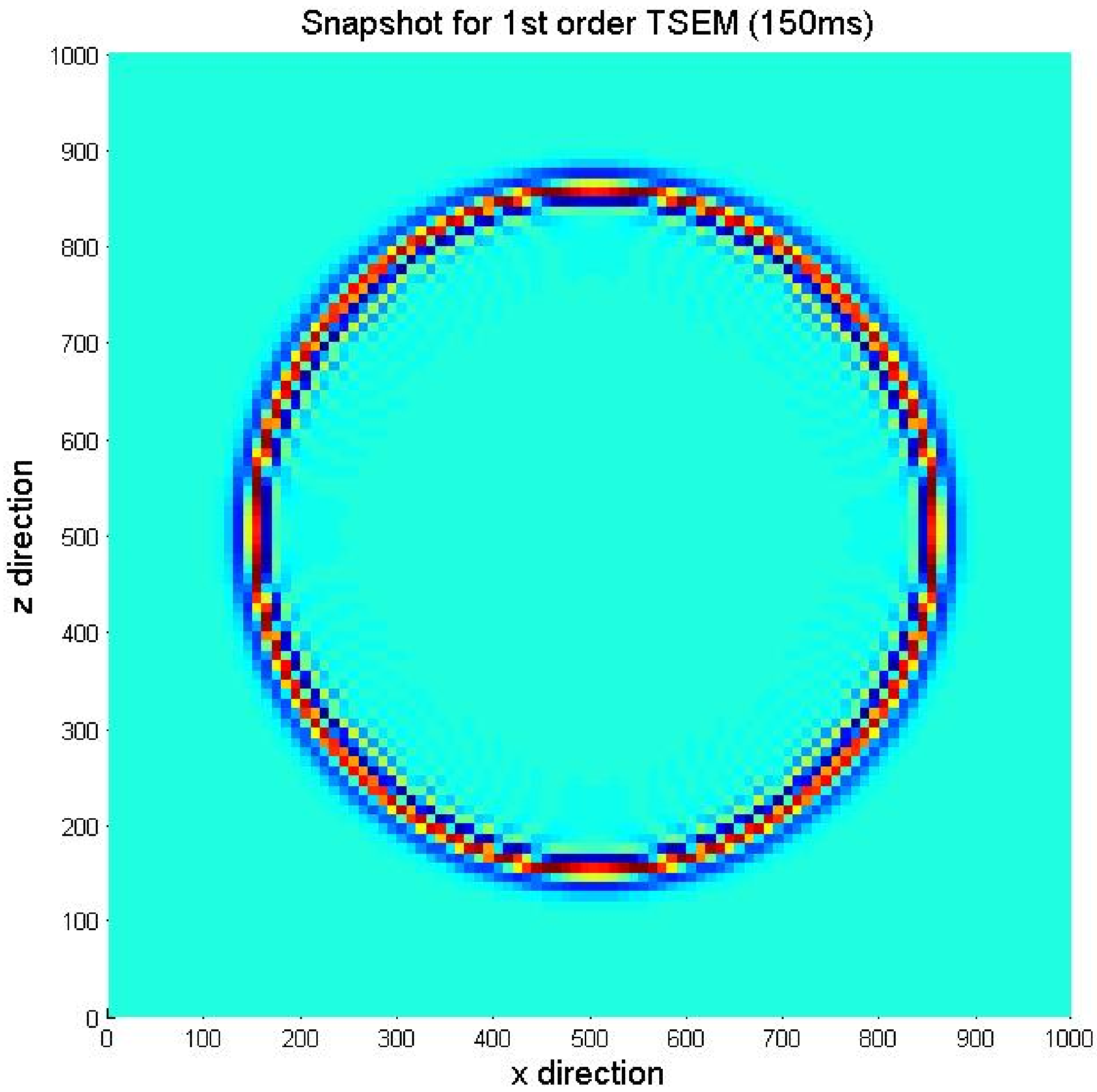
Snapshot for 3rd order (100Hz)



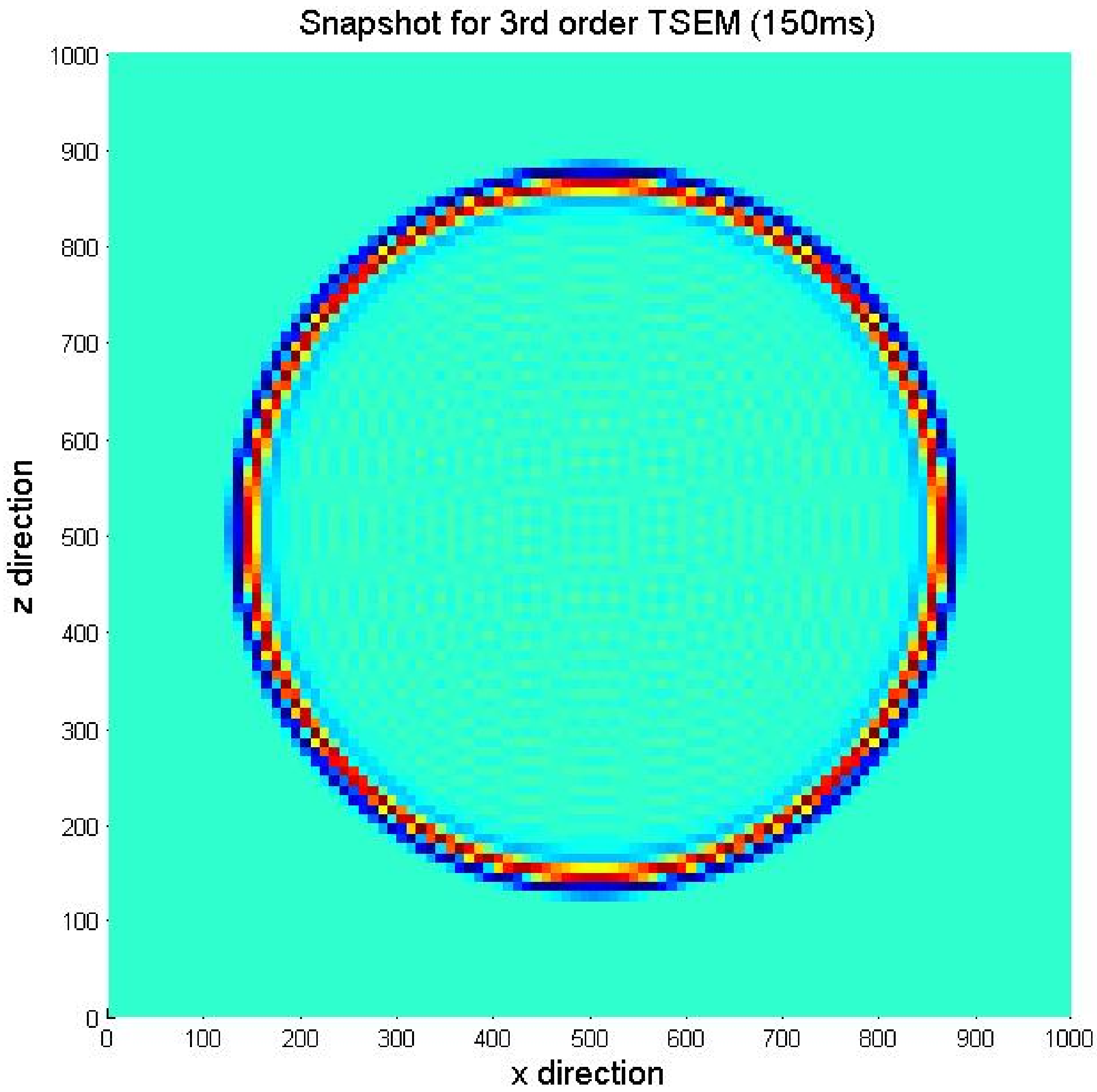
Employ one third size (100Hz)



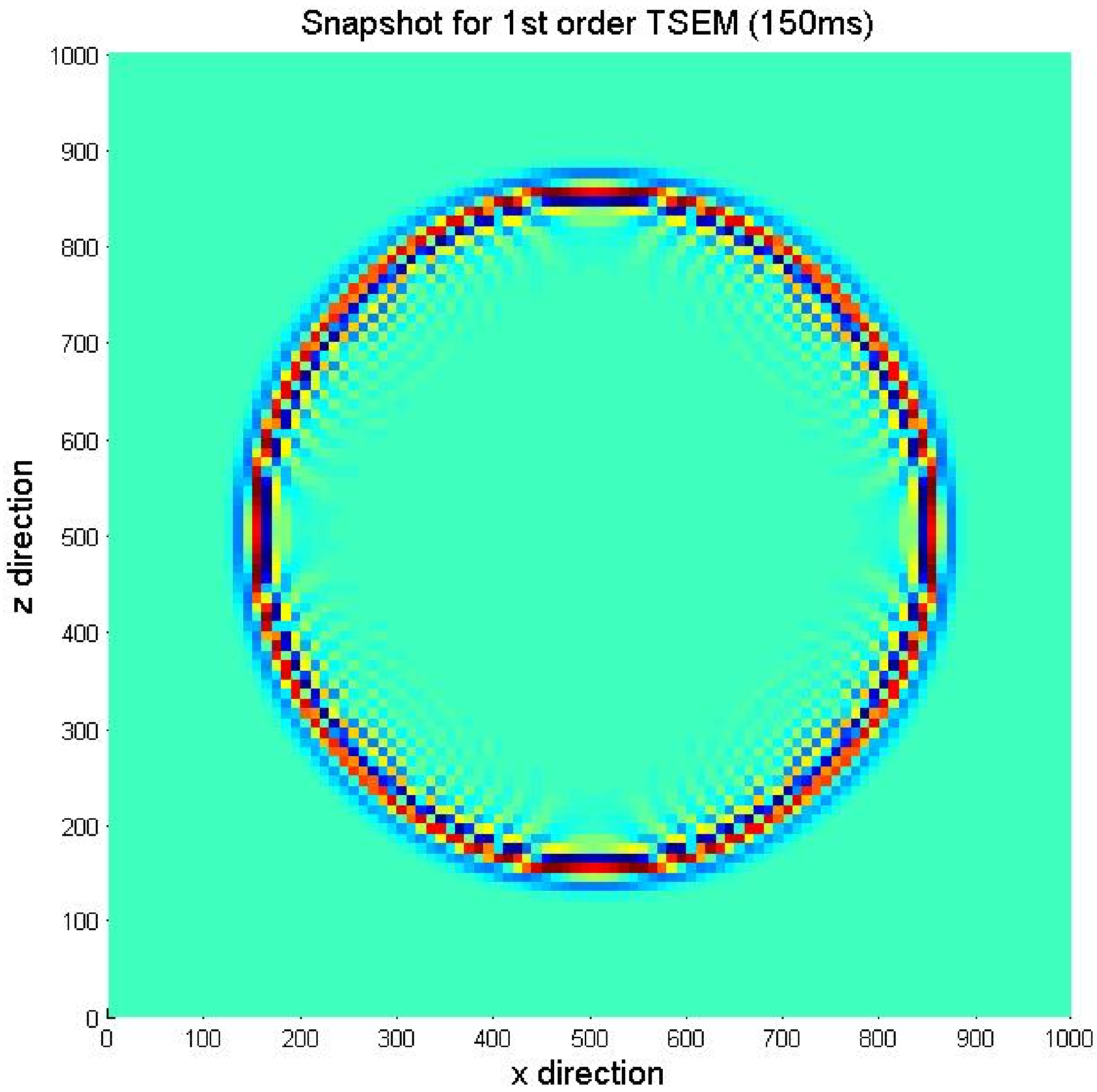
Snapshot for 1st order (100Hz)



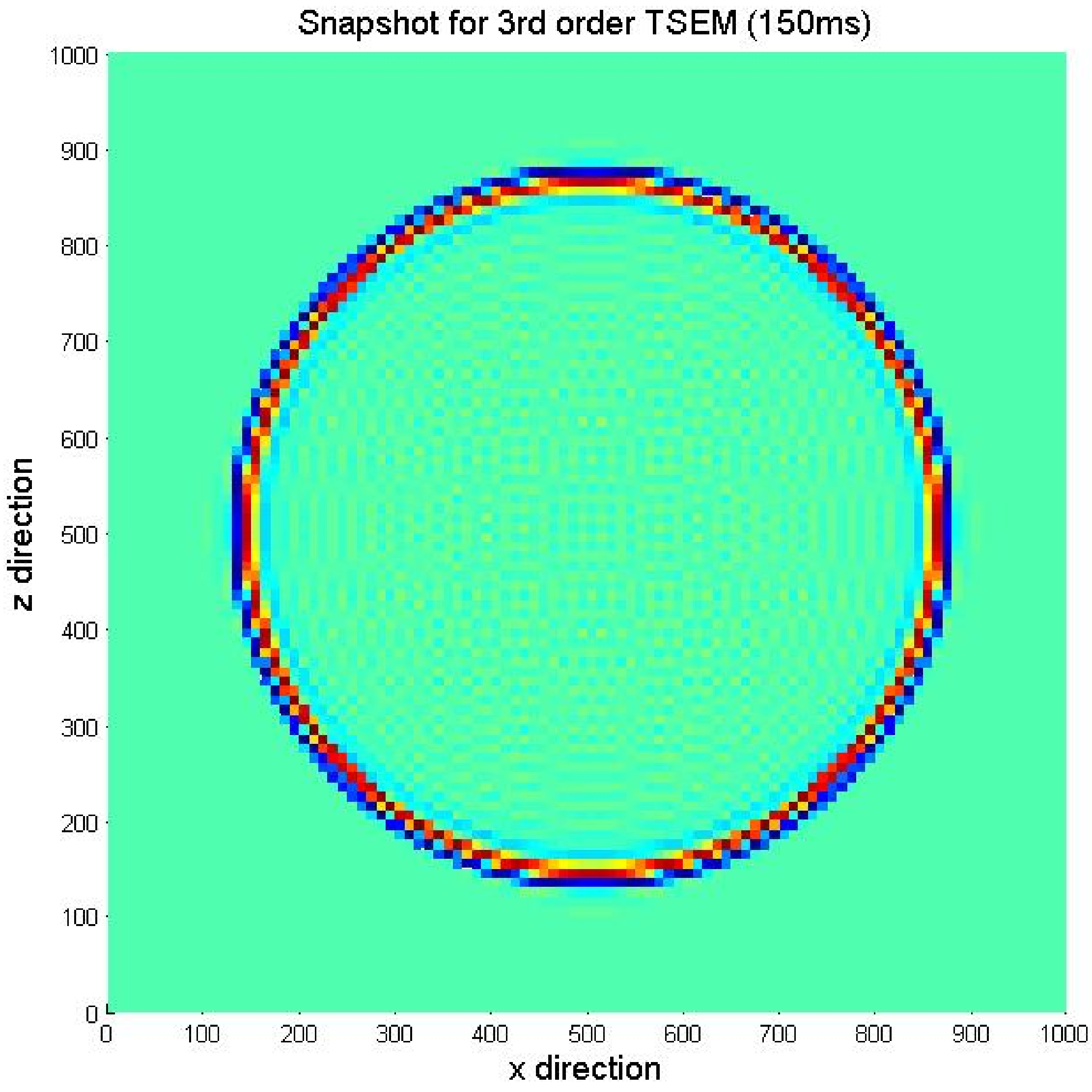
Snapshot for 3rd order (100Hz)



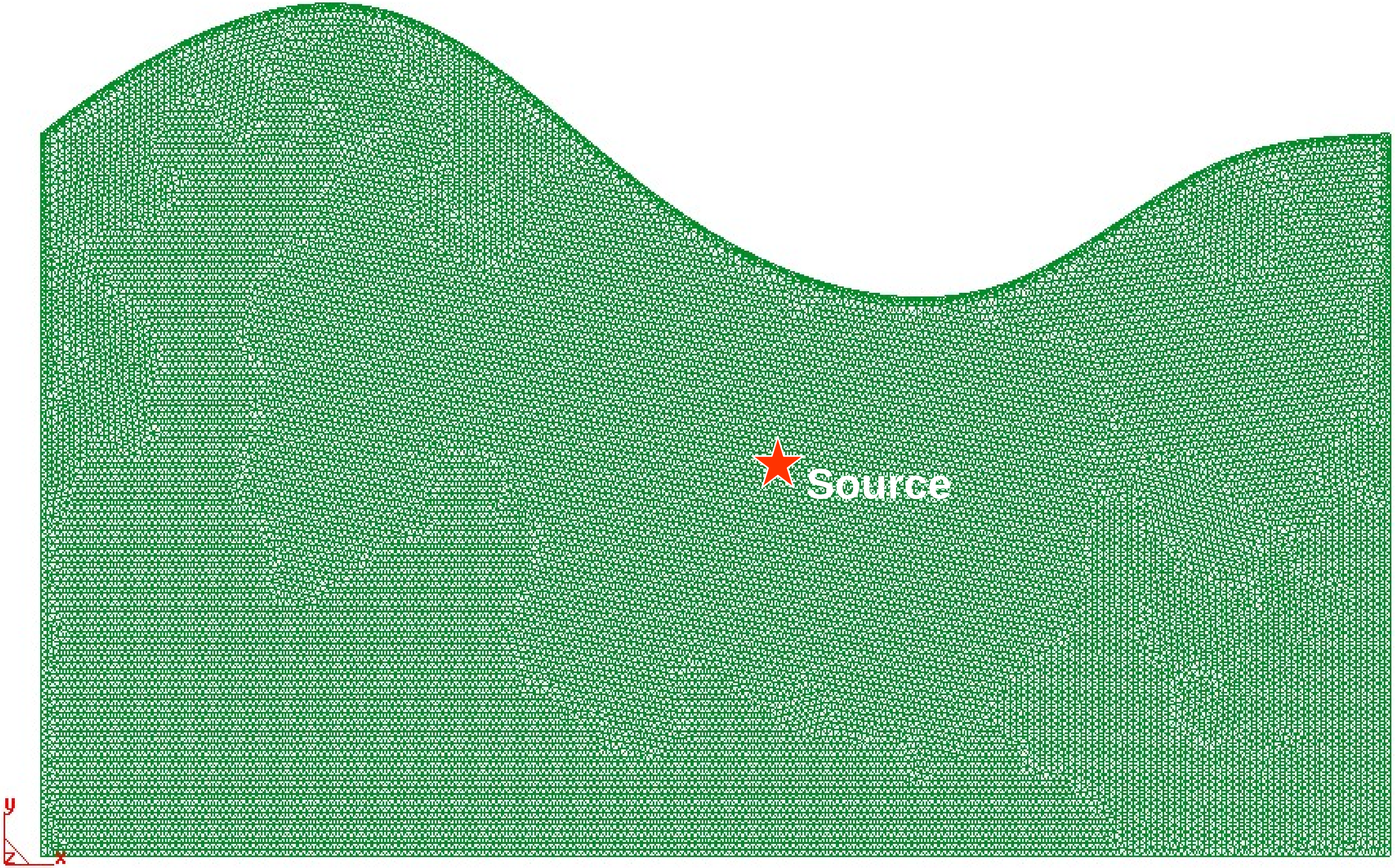
Snapshot for 1^{st} order (120Hz)



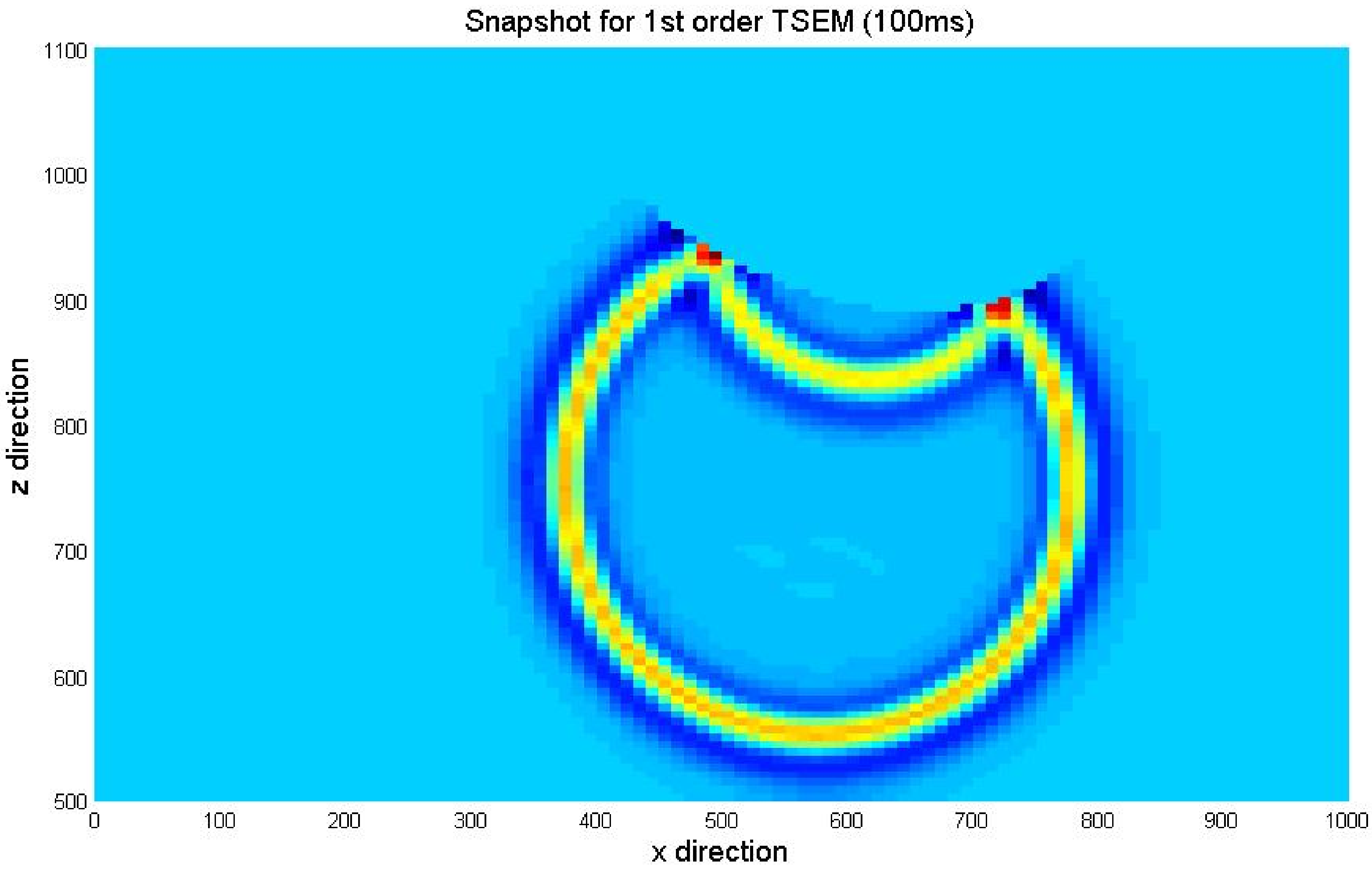
Snapshot for 3rd order (120Hz)



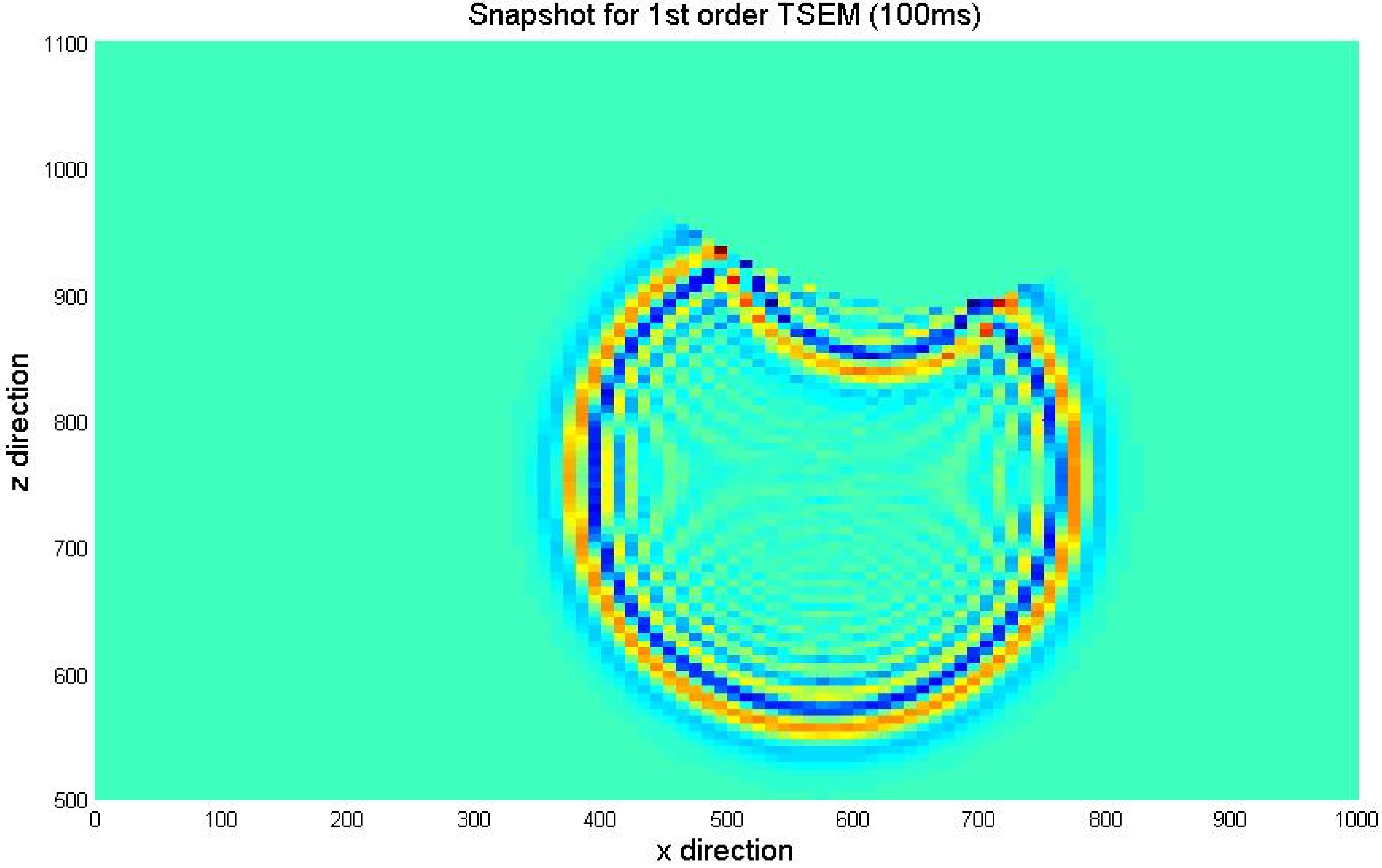
Topography problem



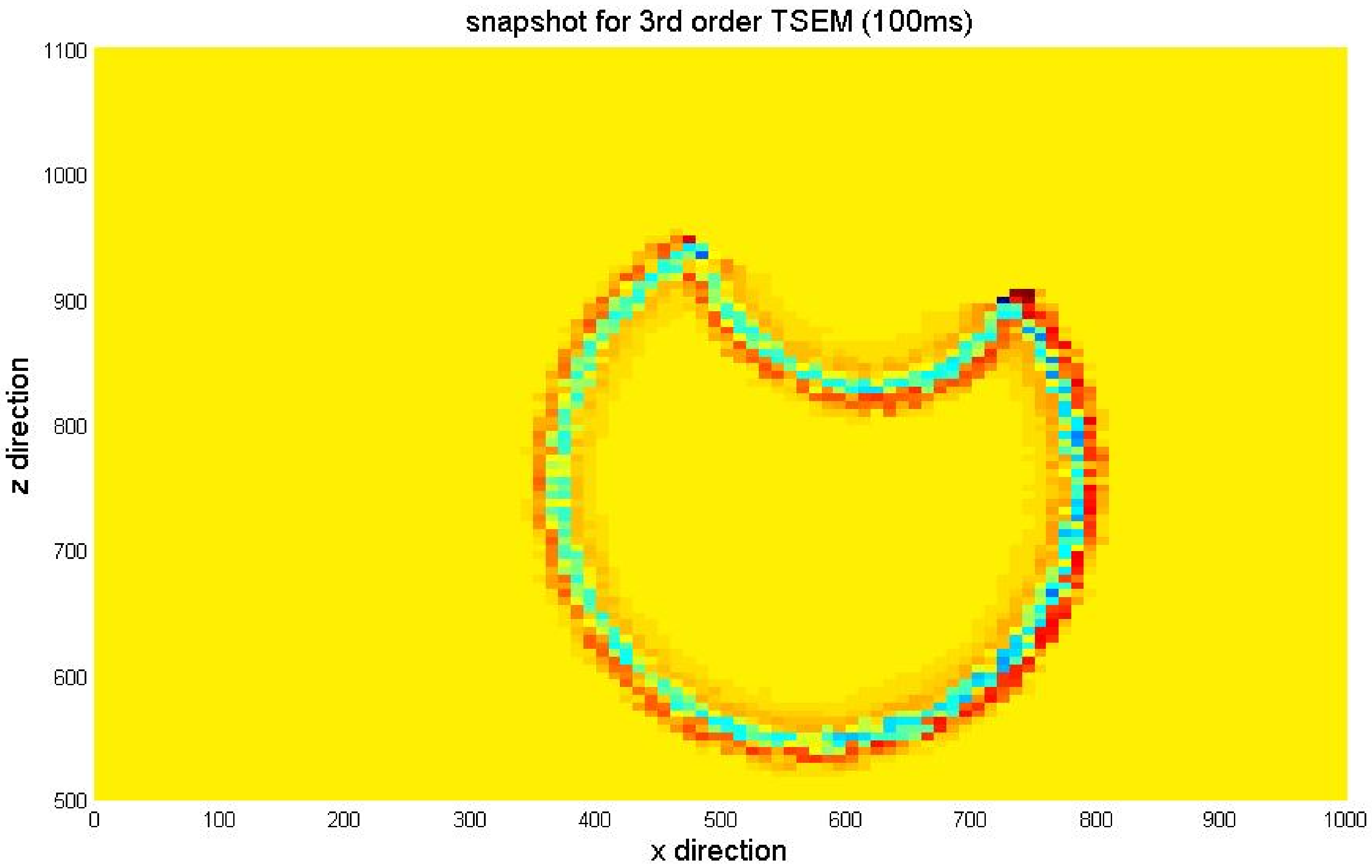
1st order result (50 Hz)



1st order result (100 Hz)



3rd order result (100 Hz)



Discussion

- Is a higher order TSEM better than a lower order TSEM?
- Consider the storage requirements
  - 1st order: each element has 3 nodes, in the stiffness matrix demands 9 element.
  - 3rd order: each element has 10 nodes, in the stiffness matrix demands 100 element, that's 10 times larger.
  - 5th order: each element has 21 nodes, in the stiffness matrix demands 441 element, that's 40 times larger.

Discussion

- The stability condition for a 3rd order TSEM is much more strict than a 1st order TSEM, which demands smaller time step.

- The interpolation function in higher order TSEM is more complicated, which may also spend a lot of time during interpretation.

- Reduction of the size of the element will confirm the accuracy for lower order TSEM, but memory consumption will increase, and the stability condition demands smaller time step.

Future Work

- Stability condition analysis
- Dispersion analysis
- Comparison with traditional methods

Acknowledgement

- Guidance of Prof. Sen
- Discussion with Dr. De Basabe
- Chinese Scholarship Council
- EDGER FORUM

Thank you

UT-AUSTIN EDGER FORUM

THE UNIVERSITY OF TEXAS AT AUSTIN
JACKSON
SCHOOL OF GEOSCIENCES

References

- **Blyth**, M. G. and Pozrikidis, C., 2006, A Lobatto interpolation grid oer the triangle. *IMA Jounral of applied Mathematics*, 71, 153~169.
- **Bos**, L., 1983. Bounding the Lebesgue function for Lagrange interpolation in a simplex, J. Approx. Theory, 38, 43~59.
- **Chen**, Q. and babuska, I., 1995. Approximate optimal points for polynomial interpolation of real functions in an interval and in a triangle. *Comput. Methods Appl. Mech. Engrg*., 128, 405~417.
- **Dubiner**, M., 1993, Spectral methods on triangles and other domains. *Journal of Scientific Computing*, 6(4), 345~390.
- **Hesthaven**, J. S., 1998. From electrostatics to almost optimal nodal sets for polynomial interpolation in a simplex. *SIAM J. Numer. Anal*, 35(2), 655-676.

THE UNIVERSITY OF TEXAS AT AUSTIN
JACKSON
SCHOOL OF GEOSCIENCES

UT-AUSTIN EDGER FORUM

- **Karniadakis E. M.** and Sherwin S., 2005. Spectral/hp element methods for computational fluid dynamics, *Oxford Science Publications*.
- **Komatitsch,** D., Martin, R., Tromp, J., Taylor, M. K. and Wingate, B. A., 2001. Wave propagation in 2-D elastic media using a spectral element method with triangles and quadrangles, *Journal of Computational Acoustics*, 9(2), 703~718.
- **Taylor**, M. A. and Wingate, B. A., 2000, An algorithm for computing Fekete points in the triangle, *SIAM J. Numer. Anal*, 38, 1708~1720.
- **Warburton**, T. C., Sherwin, S. J. and Karniadakis, G. E., 1999, Basis functions for triangular and quadrilateral high-order elements. *SIAM J. SCI. COMPUT*, 20(5), 1671~1695.

THE UNIVERSITY OF TEXAS AT AUSTIN
JACKSON
SCHOOL OF GEOSCIENCES